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# Hölder and Minkowski type inequalities for pseudo-integral

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ABSTRACT

There are proven generalizations of the Hölder's and Minkowski's inequalities for the pseudo-integral. There are considered two cases of the real semiring with pseudo-operations: one, when pseudo-operations are defined by monotone and continuous function $g$, the second semiring $([a,b], \sup, \odot)$, where $\odot$ is generated and the third semiring where both pseudo-operations are idempotent, i.e., $\oplus = \sup$ and $\odot = \inf$.



## 1. Introduction

Pseudo-analysis is a generalization of the classical analysis, where instead of the field of real numbers a semiring is defined on a real interval $[a,b] \subset [-\infty, \infty]$ with pseudo-addition $\oplus$ and with pseudo-multiplication $\odot$, see [19–24,32]. Based on this structure there were developed the concepts of $\oplus$-measure (pseudo-additive measure), pseudo-integral, pseudo-convolution, pseudo-Laplace transform, etc. The advantages of the pseudo-analysis are that there are covered with one theory, and so with unified methods, problems (usually nonlinear and under uncertainty) from many different fields (system theory, optimization, decision making, control theory, differential equations, difference equations, etc.). Pseudo-analysis uses many mathematical tools from different fields as functional equations, variational calculus, measure theory, functional analysis, optimization theory, semiring theory, etc.

Similar ideas were developed independently by Maslov and his collaborators in the framework of idempotent analysis and idempotent mathematics, with important applications [8,9,11]. In particular, idempotent analysis is fundamental for the theory of weak solutions to Hamilton–Jacobi equations with non-smooth Hamiltonians, see [8,9,11] and also [22,23,25] (in the framework of pseudo-analysis). In some cases, this theory enables one to obtain exact solutions in the similar form as for the linear equations. Some further developments relate more general pseudo-operations with applications to nonlinear partial differential equations, see [27]. Recently, these applications have become important in the field of image processing [23,25].

On the other side, more general set functions than pseudo-additive measures, as fuzzy measures and corresponding fuzzy integrals had been investigated in [6,14,21,28,31,15], as aggregation functions with important applications, e.g., given in [30,33]. Recently, there were obtained generalizations of the classical integral inequalities for integrals with respect to non-additive measures [1–4,12,16,17,26,33].

The well-known Hölder's and Minkowski's inequality is a part of the classical mathematical analysis, see [29].

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**Definition 1.** If $p$ and $g$ are positive real number such that $\frac{1}{p}+\frac{1}{q}=1$, then we call $p$ and $q$ a pair of conjugate exponents.

**Theorem 1.** *Let p and q be conjugate exponents, $1 < p < \infty$. Let X be a measure space, with measure $\mu$. Let f and g be measurable functions on X, with range in $[0,\infty]$. Then*

*(i) (Hölder's inequality)*

$$\int_X fg d\mu \leqslant \left(\int_X f^p d\mu\right)^{\frac{1}{p}}\left(\int_X g^q d\mu\right)^{\frac{1}{q}} \tag{1}$$

*(ii) (Minkowski's inequality)*

$$\left(\int_X (f+g)^p d\mu\right)^{\frac{1}{p}} \leqslant \left(\int_X f^p d\mu\right)^{\frac{1}{p}} + \left(\int_X g^p d\mu\right)^{\frac{1}{p}} \tag{2}$$

In this paper we shall give generalizations of Hölder's and Minkowski's inequality for pseudo-integral. Our paper is organized as follows. In Section 2 are given some preliminaries on the pseudo-analysis. In Section 3 special kinds of pseudo-integrals are introduced and the generated case is related to the sup-plus case. We prove generalizations of the Hölder, Minkowski for pseudo-integral in Sections 4 and 5. We note that the third important case $\oplus = \max$ and $\odot = \min$ has been studied in [1,2], where the pseudo-integral in such a case yields the Sugeno integral.

## 2. Pseudo-integral

Let $[a,b]$ be a closed (in some cases can be considered semiclosed) subinterval of $[-\infty,\infty]$. The full order on $[a,b]$ will be denoted by $\preceq$. A binary operation $\oplus$ on $[a,b]$ is pseudo-addition if it is commutative, non-decreasing (with respect to $\preceq$), associative and with a zero (neutral) element denoted by $\mathbf{0}$. Let $[a,b]_+ = \{x \mid x \in [a,b], \mathbf{0} \preceq x\}$. A binary operation $\odot$ on $[a,b]$ is pseudo-multiplication if it is commutative, positively non-decreasing, i.e., $x \preceq y$ implies $x \odot z \preceq y \odot z$ for all $z \in [a,b]_+$, associative and with a unit element $\mathbf{1} \in [a,b]$, i.e., for each $x \in [a,b]$, $\mathbf{1} \odot x = x$. We assume also $\mathbf{0} \odot x = \mathbf{0}$ and that $\odot$ is distributive over $\oplus$, i.e.,

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$$

The structure $([a,b],\oplus,\odot)$ is a *semiring* (see [10,21]). In this paper we will consider only semirings with the following continuous operations (continuity of $\odot$ can be possibly violated in the cases $\mathbf{0} \odot a = a \odot \mathbf{0}$ or $\mathbf{0} \odot b = b \odot \mathbf{0}$, i.e., in points $(\mathbf{0},a)$ and $(a,\mathbf{0})$, or $(\mathbf{0},b)$ and $(b,\mathbf{0})$), and where the boundary elements of the interval $[a,b]$ are the neutral elements of the pseudo-operations:

*Case* I: The pseudo-addition is idempotent operation and the pseudo-multiplication is not.

(a) $x \oplus y = \sup(x,y)$, $\odot$ is arbitrary not idempotent pseudo-multiplication on the interval $[a,b]$ cancellative on $]a,b[^2$. We have $\mathbf{0} = a$ and the idempotent operation supinduces a full order in the following way: $x \preceq y$ if and only if $\sup(x,y) = y$. Moreover, the pseudo-multiplication $\odot$ is generated by an increasing bijection $g : [a,b] \to [0,\infty]$, $x \odot y = g^{-1}(g(x) \cdot g(y))$ (this result follows from [7], see also [5]). Observe that $\mathbf{1} = g^{-1}(1)$.

(b) $x \oplus y = \inf(x,y)$, $\odot$ is arbitrary not idempotent pseudo-multiplication on the interval $[a,b]$ cancellative on $]a,b[^2$. We have $\mathbf{0} = b$ and the idempotent operation infinduces a full order in the following way: $x \preceq y$ if and only if $\inf(x,y) = y$. Moreover, the pseudo-multiplication $\odot$ is generated by a decreasing bijection $g : [a,b] \to [0,\infty]$, $x \odot y = g^{-1}(g(x) \cdot g(y))$ and $\mathbf{1} = g^{-1}(1)$.

*Case* II: The pseudo-operations are defined by a monotone and continuous function $g : [a,b] \to [0,\infty]$, i.e., pseudo-operations are given with

$$x \oplus y = g^{-1}(g(x) + g(y)) \text{ and } x \odot y = g^{-1}(g(x) \cdot g(y)).$$

If the zero element for the pseudo-addition is $a$, we will consider increasing generators. Then $g(a) = 0$ and $g(b) = \infty$. If the zero element for the pseudo-addition is $b$, we will consider decreasing generators. Then $g(b) = 0$ and $g(a) = \infty$.
If the generator $g$ is increasing (respectively decreasing), the operation $\oplus$induces the usual order (respectively opposite to the usual order) on the interval $[a,b]$ in the following way: $x \preceq y$ if and only if $g(x) \leqslant g(y)$.

*Case* III: Both operations are idempotent. We have

(a) $x \oplus y = \sup(x,y)$, $x \odot y = \inf(x,y)$, on the interval $[a,b]$. We have $\mathbf{0} = a$ and $\mathbf{1} = b$. The idempotent operation supinduces the usual order ($x \preceq y$ if and only if $\sup(x,y) = y$).

(b) $x \oplus y = \inf(x,y)$, $x \odot y = \sup(x,y)$, on the interval $[a,b]$. We have $\mathbf{0} = b$ and $\mathbf{1} = a$. The idempotent operation infinduces an order opposite to the usual order ($x \preceq y$ if and only if $\inf(x,y) = y$).

For $x \in [a,b]_+$ and $p \in ]0,\infty[$, we will introduce the pseudo-power $x_\odot^{(p)}$ as follows: if $p = n$ is a natural number then

$$x_\odot^{(n)} = \underbrace{x \odot x \odot \cdots \odot x}_{n}.$$

Moreover, $x_{\odot}^{(\frac{1}{n})} = \sup\{y | y_{\odot}^{(n)} \leqslant x\}$. Then $x_{\odot}^{(\frac{m}{n})} = x_{\odot}^{(r)}$ is well defined for any rational $r \in ]0, \infty[$, independently of representation $r = \frac{m}{n} = \frac{m_1}{n_1}$, $m, n, m_1, n_1$ being positive integers (the result follows from the continuity and monotonicity of $\odot$). Due to continuity of $\odot$, if $p$ is not rational, then

$$x_{\odot}^{(p)} = \sup\left\{x_{\odot}^{(r)} | r \in ]0, p[, \text{ r is rational}\right\}.$$

Evidently, if $x \odot y = g^{-1}(g(x) \cdot g(y))$, then $x_{\odot}^{(p)} = g^{-1}(g^p(x))$. On the other hand, if $\odot$ is idempotent, then $x_{\odot}^{(p)} = x$ for any $x \in [a,b]$ and $p \in ]0, \infty[$.

Let $X$ be a non-empty set. Let $\mathcal{A}$ be a $\sigma$-algebra of subsets of $X$.

**Definition 2.** A set function $m : \mathcal{A} \to [a,b]$ is a $\sigma$-$\oplus$-measure if there hold

(i) $m(\emptyset) = \mathbf{0}$ (if $\oplus$ is not idempotent)
(ii) $m(\bigcup_{i=1}^{\infty} A_i) = \bigoplus_{i=1}^{\infty} m(A_i)$ holds for any sequence $(A_i)_{i \in \mathbb{N}}$ of pairwise disjoint sets from $\mathcal{A}$.

Observe that is the case II, a set function $m : \mathcal{A} \to [a,b]$ is a $\sigma$-$\oplus$-measure if and only if $g \circ m : \mathcal{A} \to [0, \infty]$ is a classical measure, i.e., a $\sigma$-additive measure.

We suppose that $([a,b], \oplus)$ and $([a,b], \odot)$ are complete lattice ordered semigroups. We suppose that $[a,b]$ is endowed with a metric $d$ compatible with sup and inf, i.e. $\limsup x_n = x$ and $\liminf x_n = x$, imply $\lim_{n\to\infty} d(x_n, x) = 0$, and which satisfies at least one of the following conditions:

(a) $d(x \oplus y, x' \oplus y') \leqslant d(x, x') + d(y, y')$,
(b) $d(x \oplus y, x' \oplus y') \leqslant \max\{d(x, x'), d(y, y')\}$.

Both conditions (a) and (b) imply:
$d(x_n, y_n) \to 0 \Rightarrow d(x_n \oplus z, y_n \oplus z) \to 0$.

Metric $d$ is also monotonic, i.e.,
$x \leqslant z \leqslant y \Rightarrow d(x, y) \geqslant \sup\{d(y, z), d(x, z)\}$.

Let $f$ and $h$ be two functions defined on $X$ and with values in $[a,b]$. Then for any $x \in X$ and for any $\lambda \in [a,b]$ we define $(f \oplus g)(x) = f(x) \oplus g(x)$, $(f \odot g)(x) = f(x) \odot g(x)$, and $(\lambda \odot f)(x) = \lambda \odot f(x)$. The characteristic function with values in a semiring $([a,b], \oplus, \odot)$ is defined by

$$\chi_A(x) = \begin{cases} \mathbf{0}, & x \notin A, \\ \mathbf{1}, & x \in A. \end{cases}$$

A step (measurable) function is a mapping $e : X \to [a,b]$ that has the following representation $e = \bigoplus_{i=1}^{n} a_i \odot \chi_{A_i}$ for $a_i \in [a,b]$ and sets $A_i \in \mathcal{A}$ are pairwise disjoint if $\oplus$ is nonidempotent.

Let $\varepsilon$ be a positive real number and $B \subset [a,b]$. A subset $\{l_i^{\varepsilon}\}_{n \in \mathbb{N}}$ of set $B$ is a $\varepsilon$-net on $B$ if for each $x \in B$ there exists $l_i^{\varepsilon}$ such that $d(l_i^{\varepsilon}, x) \leqslant \varepsilon$. If we, also, have $l_i^{\varepsilon} \preceq x$, then we call $\{l_i^{\varepsilon}\}$ a lower $\varepsilon$-net. If $l_i^{\varepsilon} \preceq l_{i+1}^{\varepsilon}$ holds, then $\{l_i^{\varepsilon}\}$ is monotone, for more details see [13,18,21].

**Definition 3.** Let $m : \mathcal{A} \to [a,b]$ be a $\sigma$-$\oplus$-measure.

(i) The pseudo-integral of a step function $e : X \to [a,b]$ $m$ is defined by

$$\int_X^{\oplus} e \odot dm = \bigoplus_{i=1}^{n} a_i \odot m(A_i).$$

(ii) The pseudo-integral of a measurable function $f : X \to [a,b]$, (if $\oplus$ is not idempotent we suppose that for each $\varepsilon > 0$ there exists a monotone $\varepsilon$-net in $f(X)$) is defined by

$$\int_X^{\oplus} f \odot dm = \lim_{n\to\infty} \int_X^{\oplus} e_n(x) \odot dm,$$

where $(e_n)_{n \in \mathbb{N}}$ is a sequence of step functions such that $d(e_n(x), f(x)) \to 0$ uniformly as $n \to \infty$.

For more details see [9,21,23].

## 3. Explicit forms of special pseudo-integrals

We shall consider the semiring $([a,b], \oplus, \odot)$ for three (with completely different behavior) cases, namely I(a), II and III(a). Observe that the cases I(b) and III(b) are linked to the cases I(a) and III(a) by duality. First case is when pseudo-operations are generated by a monotone and continuous function $g : [a,b] \to [0, \infty]$, case II from Section 2. Then the pseudo-integral for a measurable function $f : X \to [a,b]$ is given by,

$$\int_X^{\oplus} f \odot dm = g^{-1}\left(\int_X (g \circ f) d(g \circ m)\right), \tag{3}$$

where the integral applied on the right side is the standard Lebesgue integral. In special case, when $X = [c, d]$, $\mathcal{A} = \mathcal{B}(\mathcal{X})$ and $m = g^{-1} \circ \lambda$, $\lambda$ the standard Lebesgue measure on $[c,d]$, then we use notation

$$\int_{[c,d]}^{\oplus} f(x)dx = \int_X^{\oplus} f \odot dm.$$

By (3),

$$\int_{[c,d]}^{\oplus} f(x)dx = g^{-1}\left(\int_c^d g(f(x))dx\right),$$

i.e., we have recovered the $g$-integral, see [18,19].

Second case is when the semiring is of the form $([a,b], \sup, \odot)$, cases I(a) and III(a) from Section 2. We will consider complete sup-measure $m$ only and $\mathcal{A} = 2^X$, i.e., for any system $(A_i)_{i \in I}$ of measurable sets,

$$m\left(\bigcup_{i \in I} A_i\right) = \sup_{i \in I} m(A_i).$$

Recall that if $X$ is countable (especially, if $X$ is finite) then any $\sigma$-sup-measure $m$ is complete and, moreover, $m(A) = \sup_{x \in A} \psi(x)$, where $\psi : X \to [a,b]$ is a density function given by $\psi(x) = m(\{x\})$. Then the pseudo-integral for a function $f : X \to [a,b]$ is given by

$$\int_X^{\oplus} f \odot dm = \sup_{x \in X} (f(x) \odot \psi(x)),$$

where function $\psi$ defines sup-measure $m$.

## 4. Hölder's inequality for pseudo-integral

Now we shall give a generalization of the classical Hölder inequality (1).

**Theorem 2.** *Let $p$ and $q$ be conjugate exponents, $1 < p < \infty$. For a given measurable space $(X, \mathcal{A})$ let $u, v : X \to [a,b]$ be two measurable functions and let a generator $g : [a,b] \to [0,\infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing function. Then for any $\sigma$-$\oplus$-measure $m$ it holds:*

$$\int_X^{\oplus} (u \odot v) \odot dm \leqslant \left(\int_X^{\oplus} u_{\odot}^{(p)} \odot dm\right)_{\odot}^{\left(\frac{1}{p}\right)} \odot \left(\int_X^{\oplus} v_{\odot}^{(q)} \odot dm\right)_{\odot}^{\left(\frac{1}{q}\right)}. \tag{4}$$

**Proof.** We apply the classical Hölder's inequality (1) and then we obtain

$$\int_X (g \circ u)(g \circ v) d(g \circ m) \leqslant \left(\int_X (g \circ u)^p d(g \circ m)\right)^{\frac{1}{p}} \cdot \left(\int_X (g \circ v)^q d(g \circ m)\right)^{\frac{1}{q}}.$$

Since function $g$ is increasing function, then $g^{-1}$ is also increasing function and we have

$$g^{-1}\left(\int_X (g \circ u)(g \circ v) d(g \circ m)\right) \leqslant g^{-1}\left(\left(\int_X (g \circ u)^p d(g \circ m)\right)^{\frac{1}{p}} \cdot \left(\int_X (g \circ v)^q d(g \circ m)\right)^{\frac{1}{q}}\right).$$

i.e.,

$$g^{-1}\left(\int_X g\left(g^{-1}((g \circ u)(g \circ v))\right)(g \circ m)\right) \leqslant g^{-1}\left(g\left(g^{-1}\left(\left(\int_X (g \circ u)^p d(g \circ m)\right)^{\frac{1}{p}}\right)\right) \cdot g\left(g^{-1}\left(\left(\int_X (g \circ v)^q d(g \circ m)\right)^{\frac{1}{q}}\right)\right)\right).$$

Hence

$$\int_X^{\oplus} (u \odot v) \odot dm \leqslant g^{-1}\left(\left(\int_X (g \circ u)^p d(g \circ m)\right)^{\frac{1}{p}}\right) \odot g^{-1}\left(\left(\int_X (g \circ v)^q d(g \circ m)\right)^{\frac{1}{q}}\right).$$

For the right side of the inequality holds:

$$
\begin{aligned}
&g^{-1}\left(\left(\int_X (g\circ u)^p d(g\circ m)\right)^{\frac{1}{p}}\right)\odot g^{-1}\left(\left(\int_X (g\circ v)^q d(g\circ m)\right)^{\frac{1}{q}}\right)\\
&= g^{-1}\left(\left(\int_X g\left(g^{-1}((g\circ u)^p)(g\circ m)\right)\right)^{\frac{1}{p}}\right)\odot g^{-1}\left(\left(\int_X g\left(g^{-1}((g\circ v)^q)\right)d(g\circ m)\right)^{\frac{1}{q}}\right)\\
&= g^{-1}\left(\left(\int_X g\left(u_{\odot}^{(p)}\right)d(g\circ m)\right)^{\frac{1}{p}}\right)\odot g^{-1}\left(\left(\int_X g\left(v_{\odot}^{(q)}\right)d(g\circ m)\right)^{\frac{1}{q}}\right)\\
&= g^{-1}\left(\left(g\left(g^{-1}\left(\int_X g\left(u_{\odot}^{(p)}\right)d(g\circ m)\right)\right)\right)^{\frac{1}{p}}\right)\odot g^{-1}\left(\left(g\left(g^{-1}\left(\int_X g\left(v_{\odot}^{(q)}\right)d(g\circ m)\right)\right)\right)^{\frac{1}{q}}\right)\\
&= g^{-1}\left(\left(g\left(\int_X^{\oplus} u_{\odot}^{(p)}\odot dm\right)\right)^{\frac{1}{p}}\right)\odot g^{-1}\left(\left(g\left(\int_X^{\oplus} v_{\odot}^{(q)}\odot dm\right)\right)^{\frac{1}{q}}\right) = \left(\int_X^{\oplus} u_{\odot}^{(p)}(x)dm\right)_{\odot}^{\left(\frac{1}{p}\right)}\odot\left(\int_X^{\oplus} v_{\odot}^{(q)}(x)dm\right)_{\odot}^{\left(\frac{1}{q}\right)},
\end{aligned}
$$

which completes the proof. □

**Example 1**

(i) Let $[a,b]=[0,\infty]$ and $g(x)=x^{\alpha}$ for some $\alpha\in[1,\infty[$. The corresponding pseudo-operations are $x\oplus y=\sqrt[\alpha]{x^{\alpha}+y^{\alpha}}$ and $x\odot y=xy$. Then (4) reduces on the following inequality

$$\sqrt[\alpha]{\int_{[c,d]} u(x)^{\alpha}v(x)^{\alpha}dx}\leqslant \sqrt[p\alpha]{\int_{[c,d]} u(x)^{p\alpha}dx}\,\sqrt[q\alpha]{\int_{[c,d]} v(x)^{q\alpha}dx}.$$

(ii) Let $[a,b]=[-\infty,\infty]$ and $g(x)=e^{x}$. The corresponding pseudo-operations are $x\oplus y=\ln(e^{x}+e^{y})$ and $x\odot y=x+y$. Then (4) reduces on the following inequality

$$\ln\int_{[c,d]} e^{u(x)+v(x)}dx\leqslant \frac{1}{p}\ln\left(\int_{[c,d]} e^{pu(x)}dx\right)+\frac{1}{q}\ln\left(\int_{[c,d]} e^{qv(x)}dx\right),$$

i.e.,

$$\int_{[c,d]} e^{u(x)+v(x)}dx\leqslant\left(\int_{[c,d]} e^{pu(x)}dx\right)^{\frac{1}{p}}\cdot\left(\int_{[c,d]} e^{qv(x)}dx\right)^{\frac{1}{q}}.$$

**Theorem 3.** *Let $p$ and $q$ be conjugate exponents, $1<p<\infty$. For a given measurable space $(X,\mathcal{A})$ let $u,v:X\to[a,b]$ be two measurable functions and let a generator $g:[a,b]\to[0,\infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ is a decreasing function. Then for any $\sigma$-$\oplus$-measure $m$ it holds:*

$$\int_X^{\oplus}(u\odot v)\odot dm\geqslant\left(\int_X^{\oplus} u_{\odot}^{(p)}\odot dm\right)_{\odot}^{\left(\frac{1}{p}\right)}\odot\left(\int_X^{\oplus} v_{\odot}^{(q)}\odot dm\right)_{\odot}^{\left(\frac{1}{q}\right)}.$$

**Proof.** In an analogous way as in the proof of *Theorem* (2) we obtain

$$g^{-1}\left(\int_X (g\circ u)(g\circ v)d(g\circ m)\right)\geqslant g^{-1}\left(\left(\int_X (g\circ u)^p d(g\circ m)\right)^{\frac{1}{p}}\cdot\left(\int_X (g\circ v)^q d(g\circ m)\right)^{\frac{1}{q}}\right).$$

i.e.,

$$\int_X^{\oplus}(u\odot v)\odot dm\geqslant\left(\int_X^{\oplus} u_{\odot}^{(p)}\odot dm\right)_{\odot}^{\left(\frac{1}{p}\right)}\odot\left(\int_X^{\oplus} v_{\odot}^{(q)}\odot dm\right)_{\odot}^{\left(\frac{1}{q}\right)}.\quad\square$$

Now we consider the second case, when $\oplus=\sup$, and $\odot=g^{-1}(g(x)g(y))$.

**Theorem 4.** *Let $\odot$ be represented by an increasing generator $g$ and $m$ be a complete sup-measure. Let $p$ and $q$ be conjugate exponents, $1<p<\infty$. Then for any functions $u,v:X\to[a,b]$, it holds:*

$$\int_X^{\sup} (u \odot v) \odot dm \leqslant \left( \int_X^{\sup} u_{\odot}^{(p)} \odot dm \right)_{\odot}^{\left(\frac{1}{p}\right)} \odot \left( \int_X^{\sup} v_{\odot}^{(q)} \odot dm \right)_{\odot}^{\left(\frac{1}{q}\right)}.$$

**Proof.** Recall that

$$\int_X^{\sup} (u \odot v) \odot dm = \sup_{x \in X} (u(x) \odot v(x) \odot \psi(x)) = g^{-1}\left( \sup_{x \in X} (g(u(x))g(v(x))g(\psi(x))) \right),$$

where $\psi : X \to [a,b]$ is a density function related to $m$. Moreover,

$$\left( \int_X^{\sup} u_{\odot}^{p} \odot dm \right)_{\odot}^{\left(\frac{1}{p}\right)} = g^{-1}\left( \left( \sup_{y \in X} (g^p(u(y))g(\psi(y))) \right)^{\frac{1}{p}} \right) = g^{-1}\left( \sup_{y \in X} \left( g(u(y))g^{\frac{1}{p}}(\psi(y)) \right) \right).$$

Similarly,

$$\left( \int_X^{\sup} v_{\odot}^{q} \odot dm \right)_{\odot}^{\left(\frac{1}{q}\right)} = g^{-1}\left( \sup_{z \in X} \left( g(v(z))g^{\frac{1}{q}}(\psi(z)) \right) \right).$$

Consequently,

$$\begin{aligned} \left( \int_X^{\sup} u_{\odot}^{p} \odot dm \right)_{\odot}^{\frac{1}{p}} \odot \left( \int_X^{\sup} v_{\odot}^{q} \odot dm \right)_{\odot}^{\frac{1}{q}} &= g^{-1}\left( \sup_{y \in X} \left( g(u(y))g^{\frac{1}{p}}(\psi(y)) \right) \cdot \sup_{z \in X} \left( g(v(z))g^{\frac{1}{q}}(\psi(z)) \right) \right) \\ &\geqslant g^{-1}\left( \sup_{x \in X} \left( g(u(x))g^{\frac{1}{p}}(\psi(x))g(v(x))g^{\frac{1}{q}}(\psi(x)) \right) \right) \\ &= g^{-1}\left( \sup_{x \in X} (g(u(x))g(v(x))g(\psi(x))) \right) = \int_X^{\sup} (u \odot v) \odot dm. \end{aligned}$$

Completing the proof. □

**Remark 1.** In the case III(a), i.e., if $\oplus$ = sup and $\odot$ = inf, for each $x \in [a,b]$ and $p > 0$, $x_{\odot}^{(p)} = x$. The Hölder inequality in this case reduces to the inequality

$$\int_X^{\sup} (u \odot v) \odot dm \leqslant \left( \int_X^{\sup} u \odot dm \right) \odot \left( \int_X^{\sup} v \odot dm \right),$$

i.e.,

$$\sup_{x \in X} (\inf(u(x), v(x), \psi(x))) \leqslant \inf\left( \sup_{y \in X} (\inf(u(y), \psi(y))), \sup_{z \in X} (\inf(v(z), \psi(z))) \right),$$

which trivially holds because of distributivity of sup and inf.

For a general case I(a) and III(a), i.e., when $m$ is an arbitrary $\sigma$-sup-measure, suppose that $u$ and $v$ are step function on $(X, \mathcal{A})$ with values in $[a,b]$. Then there is a finite partition $\{E_1, \ldots, E_n\}$ of $X$ so that

$$u = \sup\nolimits_{i \in \{1,2,\ldots,n\}} u_i \odot \chi_{E_i}, \quad v = \sup\nolimits_{i \in \{1,2,\ldots,n\}} v_i \odot \chi_{E_i}.$$

Define a new measurable space $(Y, 2^Y)$ with $Y = \{y_1, \ldots, y_n\}$, $y_i = E_i$, $i = 1, \ldots, n$, and define $m_Y : 2^Y \to [a,b]$, $m_Y(B) = m(\bigcup_{y_i \in B} E_i)$. Evidently, $m_Y$ is a complete sup-measure, and

$$\int_X^{\sup} u \odot dm = \int_Y^{\sup} u_Y \odot dm_Y,$$

where $u_Y(y_i) = u_i$, $i = 1, \ldots, n$. Similarly, $v_Y$ is defined and

$$\int_X^{\sup} v \odot dm = \int_Y^{\sup} v_Y \odot dm_Y.$$

Now, we are ready to prove the Hölder type theorem for general $\sigma$-sup-measure as a consequence of *Theorem* (4).

**Corollary 1.** *Let $\odot$ be represented by an increasing generator g and m be $\sigma$-sup-measure. Let p and q be conjugate exponents with $1 < p < \infty$. Then for any measurable functions $u, v : X \to [a,b]$, it holds:*

$$\int_X^{\sup} (u \odot v) \odot dm \leqslant \left( \int_X^{\sup} u_{\odot}^{(p)} \odot dm \right)_{\odot}^{\left(\frac{1}{p}\right)} \odot \left( \int_X^{\sup} v_{\odot}^{(q)} \odot dm \right)_{\odot}^{\left(\frac{1}{q}\right)}. \tag{5}$$

**Proof.** Consider the sequences $(e_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ of step function from *Definition* (3)(ii) such that $d(e_n(x),u(x)) \to 0$ and $d(f_n(x), v(x)) \to 0$ uniformly as $n \to \infty$. Then also $d((e_n \odot f_n)(x), (u \odot v)(x)) \to 0$, as well as $d\left((e_n)_{\odot}^{(p)}(x), u_{\odot}^{(p)}(x)\right) \to 0$ and $d\left((f_n)_{\odot}^{(q)}(x), v_{\odot}^{(q)}(x)\right) \to 0$ uniformly as $n \to \infty$. Due to *Theorem* (4), inequality (5) holds for each pair $(e_n,f_n)$ and thus, due to *Definition* (3)(ii), also for desired pair $(u, v)$. □

**Example 2.** For $[a,b] = [-\infty,\infty]$, let $g$ generating $\odot$ be given by $g(x) = e^x$. Then

$$x \odot y = x + y,$$

and Hölder type inequality from *Theorem* (4) reduces on

$$\sup_{x\in X}(u(x) + v(x) + \psi(x)) \leqslant \frac{1}{p} \sup_{x\in X}(p \cdot u(x) + \psi(x)) + \frac{1}{q} \sup_{x\in X}(q \cdot v(x) + \psi(x)),$$

where $u$, $v$, $\psi$ are arbitrary real function on $X$.

## 5. Minkowski's inequality for pseudo-integral

**Theorem 5.** *Let $u, v : X \to [a,b]$ be two measurable functions and $p \in [1,\infty[$. If an additive generator $g : [a,b] \to [0,\infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ are increasing. Then for any $\sigma$-$\oplus$-measure m it holds:*

$$\left( \int_X^{\oplus} (u \oplus v)_{\odot}^{(p)} \odot dm \right)_{\odot}^{\left(\frac{1}{p}\right)} \leqslant \left( \int_X^{\oplus} u_{\odot}^{(p)} \odot dm \right)_{\odot}^{\left(\frac{1}{p}\right)} \oplus \left( \int_X^{\oplus} v_{\odot}^{(p)} \odot dm \right)_{\odot}^{\left(\frac{1}{p}\right)}.$$

**Proof.** We apply the classical Minkowski's inequality (1) on compositions $g \circ u$ and $g \circ v$ and then we obtain

$$\left( \int_X (g \circ u + g \circ u)^p d(g \circ m) \right)^{\frac{1}{p}} \leqslant \left( \int_X (g \circ u)^p d(g \circ m) \right)^{\frac{1}{p}} + \left( \int_X (g \circ v)^p d(g \circ m) \right)^{\frac{1}{p}}.$$

Since function $g$ is increasing function, then $g^{-1}$ is also increasing function and we have

$$g^{-1}\left( \left( \int_X (g \circ u + g \circ u)^p d(g \circ m) \right)^{\frac{1}{p}} \right) \leqslant g^{-1}\left( \left( \int_X (g \circ u)^p d(g \circ m) \right)^{\frac{1}{p}} + \left( \int_X (g \circ v)^p d(g \circ m) \right)^{\frac{1}{p}} \right).$$

Hence

$$\begin{aligned}
g^{-1}\left( \left( \int_X (g \circ u + g \circ v)^p d(g \circ m) \right)^{\frac{1}{p}} \right) &= g^{-1}\left( \left( \int_X \left(g(g^{-1}(g \circ u + g \circ v))\right)^p d(g \circ m) \right)^{\frac{1}{p}} \right) \\
&= g^{-1}\left( \left( \int_X g\left(g^{-1}\left( (g(g^{-1}(g \circ u + g \circ v)))^p \right)\right) d(g \circ m) \right)^{\frac{1}{p}} \right) \\
&= g^{-1}\left( \left( \int_X g\left(g^{-1}\left((g(u \oplus v))^p\right)\right) d(g \circ m) \right)^{\frac{1}{p}} \right) \\
&= g^{-1}\left( \left( \int_X g\left((u \oplus v)_{\odot}^{(p)}\right) d(g \circ m) \right)^{\frac{1}{p}} \right) \\
&= g^{-1}\left( \left( g\left(g^{-1}\left( \int_X g\left((u \oplus v)_{\odot}^{(p)}\right) d(g \circ m) \right)\right) \right)^{\frac{1}{p}} \right) = \left( \int_X^{\oplus} (u \oplus v)_{\odot}^{(p)} \odot dm \right)_{\odot}^{\left(\frac{1}{p}\right)}.
\end{aligned}$$

On the other side, we have

$$g^{-1}\left(\left(\int_X (g\circ u)^p d(g\circ m)\right)^{\frac{1}{p}} + \left(\int_X (g\circ v)^p d(g\circ m)\right)^{\frac{1}{p}}\right) = g^{-1}\left(g\left(g^{-1}\left(\left(\int_X g(g^{-1}((g\circ u)^p)d(g\circ m))\right)^{\frac{1}{p}}\right)\right)\right.$$
$$\left. + g\left(g^{-1}\left(\left(\int_X g(g^{-1}((g\circ v)^p))d(g\circ m)\right)^{\frac{1}{p}}\right)\right)\right)$$
$$= g^{-1}\left(\left(\int_X g\left(u_{\odot}^{(p)}\right)d(g\circ m)\right)^{\frac{1}{p}}\right)$$
$$\oplus g^{-1}\left(\left(\int_X g\left(v_{\odot}^{(p)}(x)\right)d(g\circ m)\right)^{\frac{1}{p}}\right)$$
$$= g^{-1}\left(\left(g\left(\int_X^{\oplus} u_{\odot}^{(p)} \oplus dm\right)\right)^{\frac{1}{p}}\right) \oplus g^{-1}\left(\left(g\left(\int_X^{\oplus} v_{\odot}^{(p)} \odot dm\right)\right)^{\frac{1}{p}}\right)$$
$$= \left(\int_X^{\oplus} u_{\odot}^{(p)} \odot dm\right)_{\odot}^{\left(\frac{1}{p}\right)} \oplus \left(\int_X^{\oplus} v_{\odot}^{(q)} \odot dm\right)_{\odot}^{\left(\frac{1}{q}\right)},$$

which completes the proof. □

In the case of idempotent pseudo-addition $\oplus = \sup$ also a version of Minkowski inequality holds. Observe that if $\odot = \inf$, then the corresponding inequality means (recall that $x_{\odot}^{(p)} = x$ for each $x \in [a,b]$, $p > 0$)

$$\int_X^{\oplus} (u \oplus v) \odot dm \leqslant \sup\left(\int_X^{\oplus} u \odot dm, \int_X^{\oplus} v \odot dm\right),$$

i.e.,

$$\sup_{x\in X} \inf(\sup(u(x), v(x)), \psi(x)) \leqslant \sup\left(\sup_{y\in X}(\inf(u(y), \psi(y))), \sup_{z\in X}(\inf(v(z), \psi(z)))\right),$$

which holds due to the distributivity of sup and inf.

Finally, we turn our attention to the case I(a).

**Theorem 6.** *Let $\odot$ be represented by an increasing generator $g$, $m$ be a complete sup-measure and $p \in ]0,\infty[$. Then for any functions $u, v: X \to [a,b]$, it holds:*

$$\left(\int_X^{\sup} (\sup(u,v))_{\odot}^{(p)} \odot dm\right)_{\odot}^{\left(\frac{1}{p}\right)} = \sup\left(\left(\int_X^{\sup} u_{\odot}^{(p)} \odot dm\right)_{\odot}^{\left(\frac{1}{p}\right)}, \left(\int_X^{\sup} v_{\odot}^{(p)} \odot dm\right)_{\odot}^{\left(\frac{1}{p}\right)}\right).$$

**Proof.** Let $\psi$ be the density function related to $m$. As already was shown in proof of *Theorem* (4),

$$\left(\int_X^{\sup} u_{\odot}^{p} \odot dm\right)^{\left(\frac{1}{p}\right)} = g^{-1}\left(\sup_{y\in X}\left(g(u(y))g^{\frac{1}{p}}(\psi(y))\right)\right),$$

and

$$\left(\int_X^{\sup} v_{\odot}^{p} \odot dm\right)^{\left(\frac{1}{p}\right)} = g^{-1}\left(\sup_{z\in X}\left(g(v(z))g^{\frac{1}{p}}(\psi(z))\right)\right).$$

Therefore

$$\sup\left(\left(\int_X^{\sup} u_{\odot}^{(p)} \odot dm\right)_{\odot}^{\left(\frac{1}{p}\right)}, \left(\int_X^{\sup} v_{\odot}^{(p)} \odot dm\right)_{\odot}^{\left(\frac{1}{p}\right)}\right) = g^{-1}\left(\sup\left(\sup_{y\in X}\left(g(u(y))g^{\frac{1}{p}}(\psi(y))\right), \sup_{z\in X}\left(g(v(z))g^{\frac{1}{p}}(\psi(z))\right)\right)\right)$$
$$= g^{-1}\left(\sup_{x\in X}\left(g(\sup(u(x), v(x)))\cdot g^{\frac{1}{p}}(\psi(x))\right)\right)$$
$$= \left(\int_X^{\oplus} (\sup(u,v))_{\odot}^{(p)} \odot dm\right)_{\odot}^{\left(\frac{1}{p}\right)},$$

what is the result we have to prove. □

## 6. Conclusion

We have proved the Hölder and Minkowski integral type inequality for the pseudo-integral for its characteristic cases. There are several classical inequalities related to the Lebesgue integral which can be generated for non-linear integrals, including the pseudo-integrals. Up to the published results mentioned in the introduction, recall, for example, the Markov inequality

$$m(f \geqslant c) \leqslant \frac{1}{c}\int_X f dm,$$

valid for any integrable non-negative function $f$ and positive constant $c$. Rewriting the Markov inequality into its equivalent form

$$c \cdot m(f \geqslant c) \leqslant \int_X f dm,$$

we have a straightforward generalization for pseudo-integrals, namely

$$c \odot m(f \geqslant c) \leqslant \int_X^{\oplus} f \odot dm, \tag{6}$$

whenever $c \in [a,b]_+ \backslash \{\mathbf{0}\}$ and $f : X \to [a,b]_+$.

This inequality is trivial if $\oplus$induced the standard ordering on $[a,b]$, i.e., in cases I(a), III(a) and II with increasing generator $g$, due to the non-decreasingness of the corresponding pseudo-integrals, see [18,21], and the fact that the function $f_c : X \to [a,b]$ given by

$$f_c(x) = \begin{cases} c, & \text{if } f(x) \geqslant c, \\ \mathbf{0}, & \text{else} \end{cases}$$

satisfies $f_c \leqslant f$, and

$$\int_X^{\oplus} f \odot dm = c \odot m(f \geqslant c).$$

Moreover, if $\odot$ is cancellative on $]a,b[$ (see cases I(a) and II with $g$ increasing), then (6) generalized into

$$c_{\odot}^{(p)} \odot m(f \geqslant c) \leqslant \int_X^{\oplus} f_{\odot}^{(p)} \odot dm,$$

where $p$ is an arbitrary positive constant (recall the case $p = 2$ for the classical Lebesgue integral, which is a version of Chebyshev inequality in statistics).

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